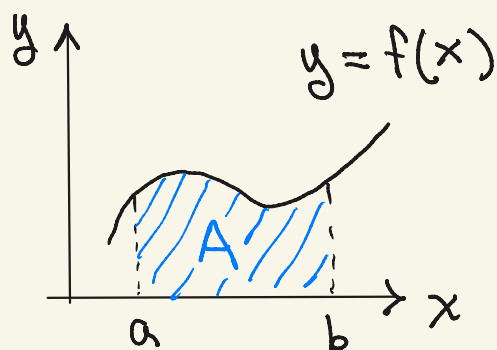


§ 15.1 Double Integrals -

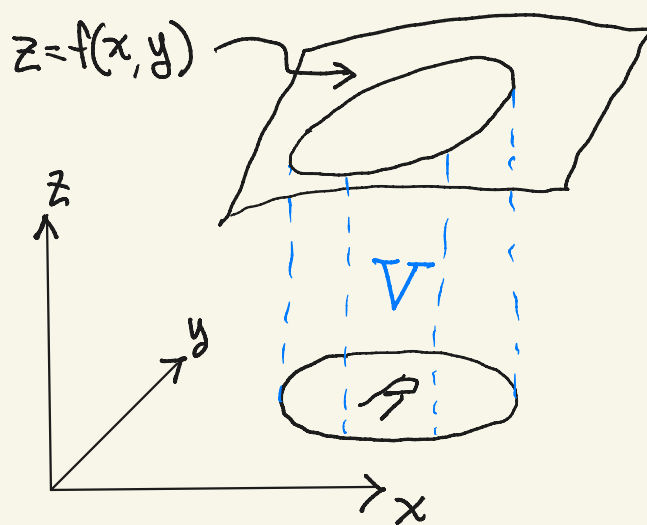
- Defn in Words: $\iint_R f(x,y) dA \equiv$ "The volume of the region under the graph of f above the (x,y) -plane."

- Generalizes: $\int_a^b f(x) dx \equiv$ "The area of the region under the graph of f above the interval $[a,b]$."

- Picture



$$A = \int_a^b f(x) dx$$



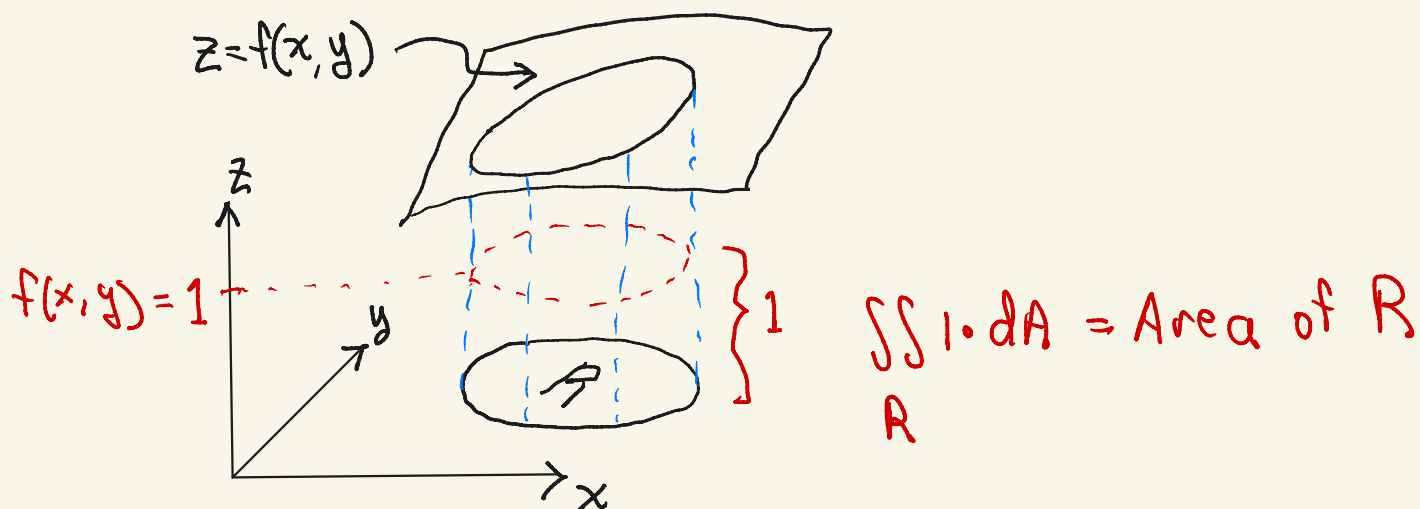
$$V = \iint_R f(x,y) dA$$

- Note: If $f(x,y) = 1$, then

$$\iint_R 1 \cdot dA = \text{Area of } R$$

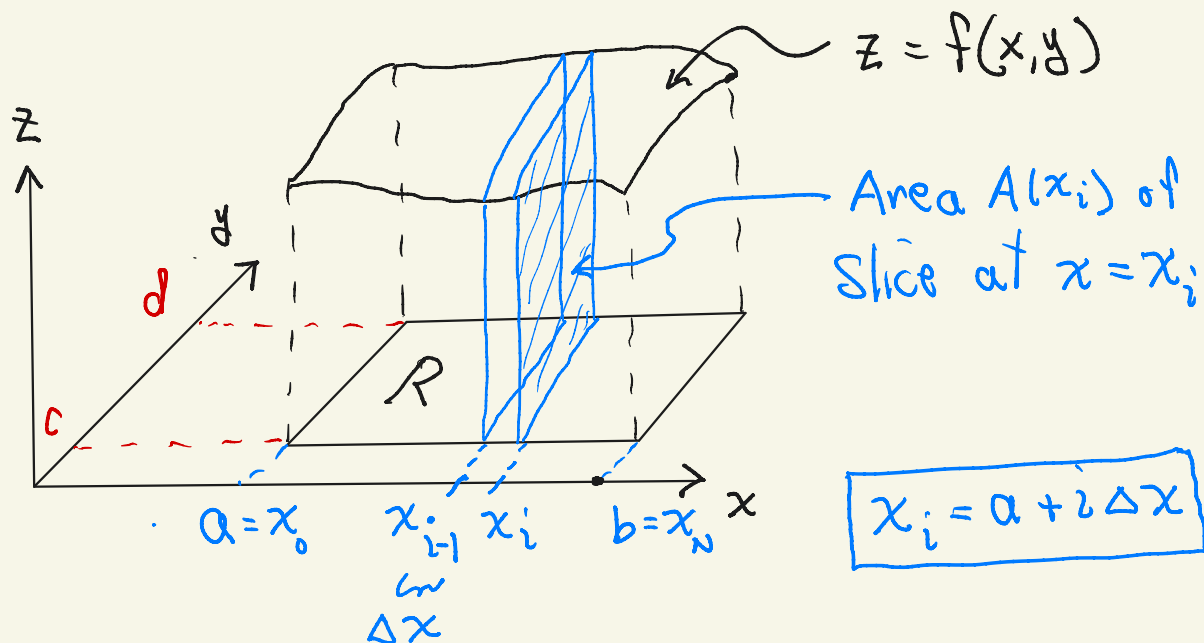
that is... if $f(x,y)=1$

2



Simplest Case: R is a Rectangle $R=[a,b] \times [c,d]$

Notation: $R=[a,b] \times [c,d] = \{(x,y) : x \in [a,b] \text{ \& } y \in [c,d]\}$

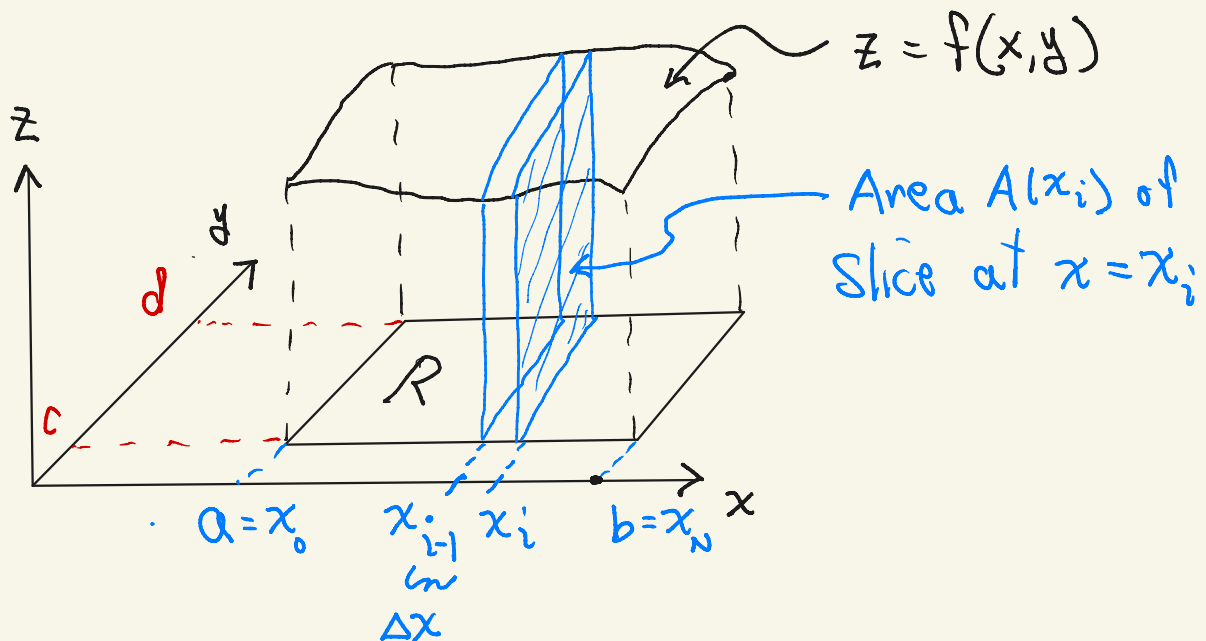


Define a mesh: $a=x_0 < x_1 < \dots < x_i < \dots < x_N=b$

Always the same way: $N = \#$ of mesh points

$\Delta x = \text{dist betw points} = x_i - x_{i-1} = \frac{b-a}{N}$, $x_i = a + i \Delta x$

- Now... construct the volume from an approximate volume based on slices of Area



$$\iint_R f(x, y) dA = \text{"Vol" under graph of } f \text{ above } \begin{matrix} d \\ c \end{matrix} \begin{matrix} a & b \end{matrix} \begin{matrix} P \\ R \end{matrix}$$

$$\text{Let } A(x) = \text{"Area of slice at } x \text{"} = \int_c^d f(x, y) dy$$

$$\text{Vol} \approx \underbrace{\sum_{i=1}^N A(x_i) \Delta x}_{\text{Riemann Sum}} = \int_a^b A(x) dx = \int_a^b \underbrace{\int_c^d f(x, y) dy}_{A(x)} dx$$

Conclude: $\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx$

Integrate wrt y at fixed x to get a function of x alone

Theorem : (Fubini - I) When R is a rectangle (4)

$$R = [a, b] \times [c, d]$$

$$a \leq x \leq b$$

$$c \leq y \leq d$$

we have :

$$\int_a^b \int_c^d f(x, y) dy dx = \iint_R f(x, y) dA = \int_c^d \int_a^b f(x, y) dx dy$$

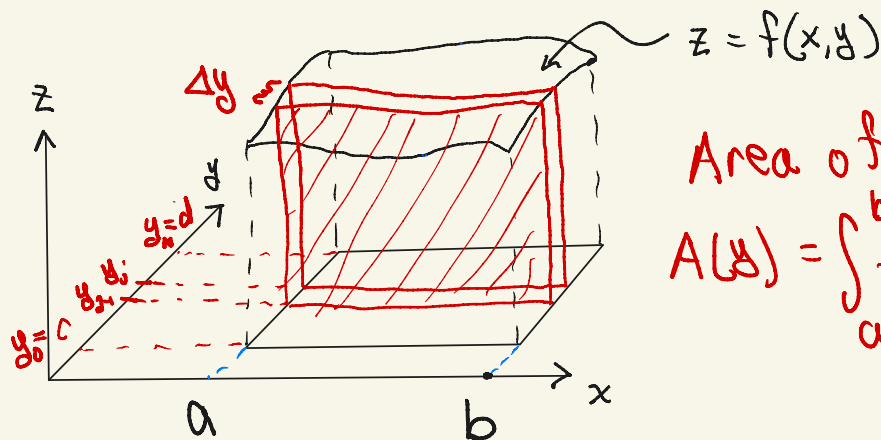
Proof : Put the mesh on y -axis

$$c = y_0 < y_1 < \dots < y_j < \dots < y_N = d$$

$$\Delta y = \frac{d-c}{N}, \quad y_j = c + j \Delta y$$

$$\begin{aligned} \text{So } \iint_R f(x, y) dA &\approx \sum_{j=1}^N A(y_j) \Delta y \xrightarrow{\Delta y \rightarrow 0} \int_c^d A(y) dy \\ &= \int_c^d \underbrace{\int_a^b f(x, y) dx}_{A(y)} dy \end{aligned}$$

Picture



Area of slice

$$A(y) = \int_a^b f(x, y) dx$$

Example: $f(x,y) = xy$, $0 \leq x \leq 1$, $1 \leq y \leq 2$

(5)

$R = [0,1] \times [1,2]$ evaluate $I = \iint_R f(x,y) dA$ two ways -

Solution:

$$\begin{aligned} \textcircled{1} \quad I &= \int_0^1 \int_1^2 xy \, dy \, dx = \int_0^1 x \left[\frac{y^2}{2} \right]_{y=1}^{y=2} dx = \int_0^1 x \left(\frac{4}{2} - \frac{1}{2} \right) dx \\ &= \frac{3}{2} \int_0^1 x \, dx = \frac{3}{2} \left[\frac{x^2}{2} \right]_0^1 = \frac{3}{2} \cdot \frac{1}{2} = \boxed{\frac{3}{4}} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad I &= \int_1^2 \int_0^1 xy \, dx \, dy = \int_1^2 \left[\frac{x^2}{2} y \right]_{x=0}^{x=1} dy = \int_1^2 \frac{1}{2} y \, dy \\ &= \frac{1}{2} \left[\frac{y^2}{2} \right]_1^2 = \frac{1}{2} \left(\frac{2^2}{2} - \frac{1^2}{2} \right) = \boxed{\frac{3}{4}} \end{aligned}$$

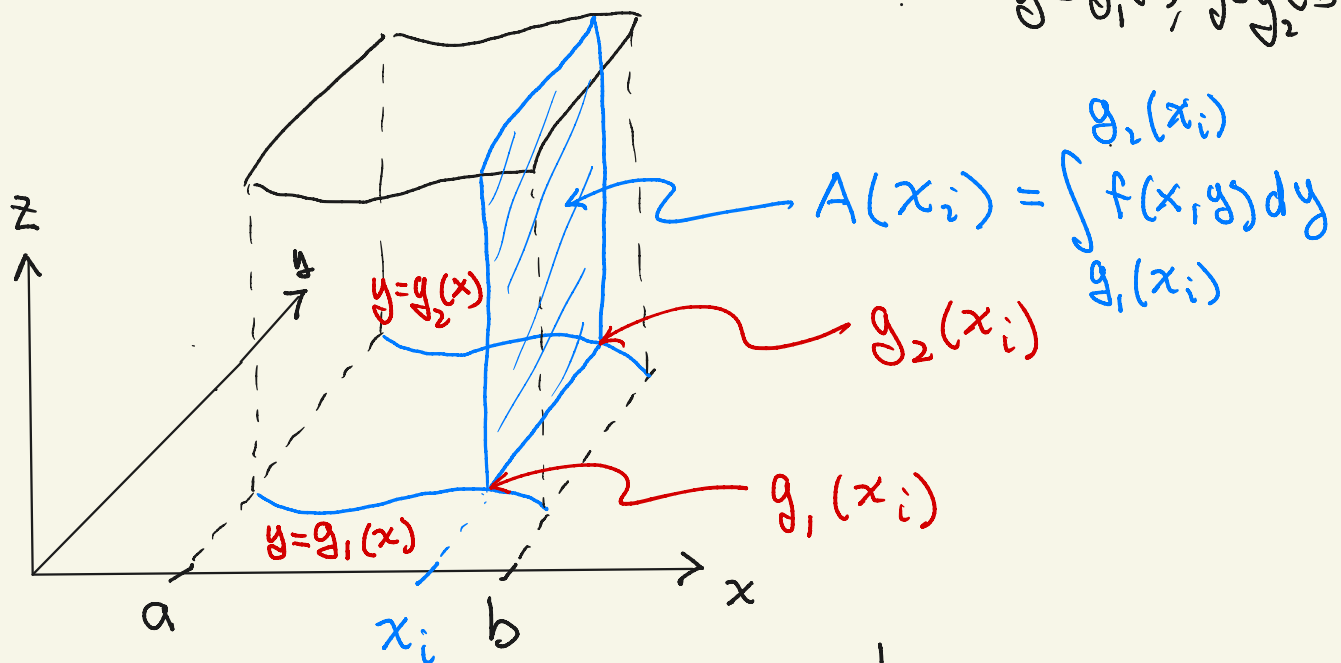
Note: Both iterated integrals are solved by different arithmetic but yield same ans!

Note: In general, one iterate may be solvable by Math 21B methods, when the other might not!

When R is not a rectangle, the procedure for changing the order of integration is more complicated!

General Case of General Slicing

$R \equiv$ region in (x, y) -plane bounded by $x = a, x = b$
 $y = g_1(x), y = g_2(x)$



$$\iint_R f(x, y) dA \approx \underbrace{\sum_{i=1}^N A(x_i) \Delta x}_{\text{Riemann Sum}} \xrightarrow{N \rightarrow \infty} \int_a^b \underbrace{A(x)}_{\int_{g_1(x)}^{g_2(x)} f(x, y) dy} dx$$

Note: You might not be able to slice the other way!

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$$

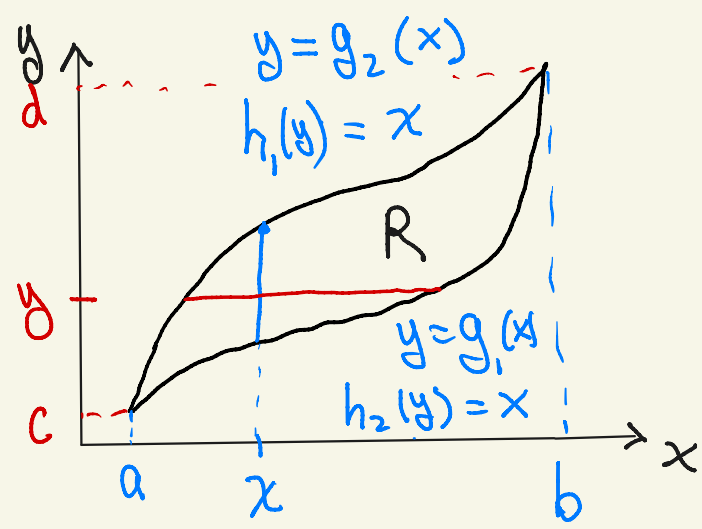
This is a formula for computing the volume

General Case of Fubini's Thm when you can iterate an integral both ways

To iterate, you only need R in (x,y) -plane

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y) dy$$

$$A(y) = \int_{h_1(y)}^{h_2(y)} f(x,y) dx$$



Then: $I = \iint_R f(x,y) dA$ can be iterated 2-ways: (sliced)

$$I = \int_a^b A(x) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$

$$I = \int_c^d A(y) dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy$$

Recall: Riemann Integral requires f be continuous

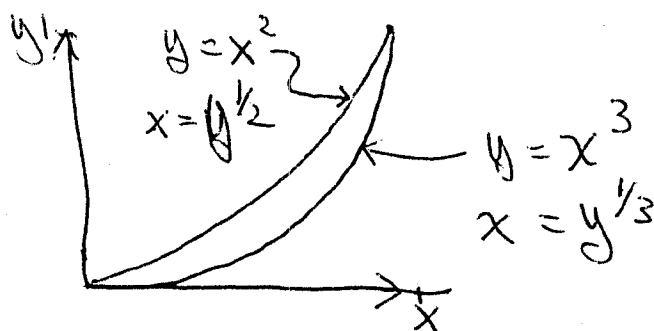
Note: $\boxed{\begin{matrix} y = g_2(x) \\ h_1(y) = x \end{matrix}} \Leftrightarrow g_2^{-1} = h_1$ $\boxed{\begin{matrix} y = g_1(x) \\ h_2(y) = x \end{matrix}} \Leftrightarrow g_1^{-1} = h_2$

(6)

Ex: Evaluate $\iint_R x \, dA$ where R is

region between ^{curves} ~~graphs~~ of $y = x^3$ & $y = x^2$ 2-ways

Picture:



$$\textcircled{1} \int_0^1 \int_{x^3}^{x^2} x \, dy \, dx = \int_0^1 x y \Big|_{y=x^3}^{y=x^2} dx = \int_0^1 x^3 - x^4 \, dx = \left[\frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

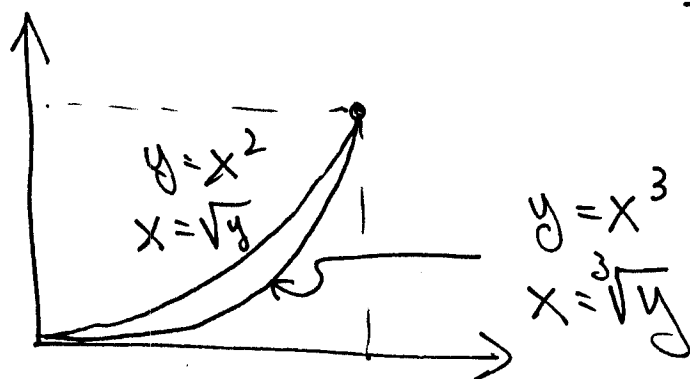
$$\begin{aligned} \textcircled{2} \int_0^1 \int_{\sqrt[3]{y}}^{\sqrt[2]{y}} y \, dx \, dy &= \int_0^1 \left[\frac{x^2}{2} \right]_{x=\sqrt[3]{y}}^{x=\sqrt[2]{y}} dy = \int_0^1 \frac{(\sqrt[2]{y})^2}{2} - \frac{(\sqrt[3]{y})^2}{2} dy \\ &= \int_0^1 \frac{y^{2/3}}{2} - \frac{y^{2/3}}{2} dy = \left[\frac{3}{5} \frac{y^{5/3}}{2} - \frac{y^2}{4} \right]_0^1 = \frac{3}{10} - \frac{1}{4} \\ &= \frac{12-10}{40} = \frac{1}{20} \end{aligned}$$

Use dbl Int

Ex: Find the area between the

curves $y = x^3$ and $y = x^2$ 3-ways

Picture:



$$\textcircled{1} \int_0^1 \int_{x^3}^{x^2} 1 \cdot dy dx = \int_0^1 [x^2 - x^3] dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}$$

$$\textcircled{2} \int_0^1 \int_{\sqrt{y}}^{\sqrt[3]{y}} 1 \cdot dx dy = \int_0^1 [y^{1/3} - y^{1/2}] dy = \left[\frac{3y^{4/3}}{4} - \frac{2y^{3/2}}{3} \right]_0^1 = \frac{1}{12}$$

3) Check from 2/A: Area betw f & g

$$\text{Area} = \int_a^b [f(x) - g(x)] dx = \int_0^1 x^2 - x^3 dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \checkmark$$

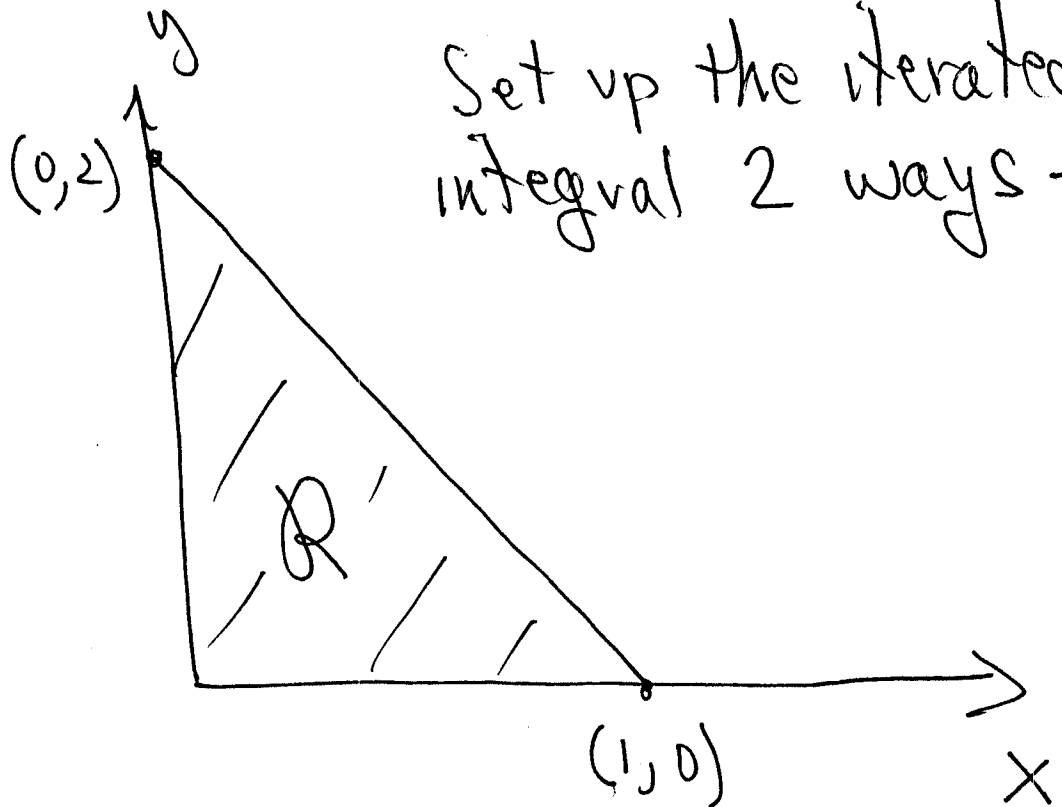


Ex Find the volume under the graph^⑧ of $f(x,y) = xy$ above the region R bounded by the x -axis, y -axis & the line that passes thru $(1,0)$ & $(0,2)$

Picture:

(2-ways)

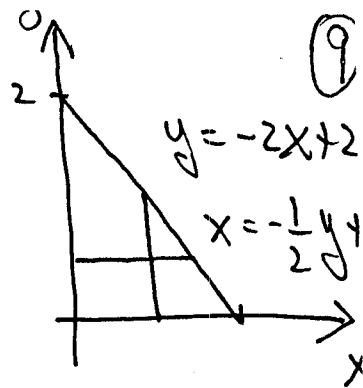
Set up the iterated integral 2 ways -



Soln

$$y = -2x + 2, \quad 2x = -y + 2$$

$$x = -\frac{1}{2}y + 1$$



$$\int_0^1 \int_0^{-2x+2} xy \, dy \, dx = \int_0^2 \int_0^{-\frac{1}{2}y+1} xy \, dx \, dy$$

$$\text{LHS} = \int_0^1 x \left[\frac{y^2}{2} \right]_0^{-2x+2} dx = \int_0^1 x \left[\frac{(-2x+2)^2}{2} - 0 \right] dx$$

$$= \int_0^1 x \left[\frac{4x^2 + (-8x) + 4}{2} \right] dx = \int_0^1 (2x^3 - 4x^2 + 2x) dx$$

$$= 2 \left[\frac{x^4}{4} - 4 \frac{x^3}{3} + 2 \frac{x^2}{2} \right]_0^1 = \frac{2}{4} - \frac{4}{3} + \frac{2}{2} = \frac{3}{2} - \frac{4}{3} = \frac{1}{6}$$

$$\text{RHS} = \int_0^2 \int_0^{-\frac{1}{2}y+1} xy \, dx \, dy = \int_0^2 y \int_0^{-\frac{1}{2}y+1} x \, dx \, dy$$

$$= \int_0^2 y \left[\frac{x^2}{2} \right]_0^{-\frac{1}{2}y+1} dy = \int_0^2 y \left[\frac{(-\frac{1}{2}y+1)^2}{2} \right] dy = \int_0^2 y \left[\frac{+\frac{1}{4}y^2 - y + 1}{2} \right] dy$$

$$= \int_0^2 \left[\frac{1}{8}y^3 - \frac{y^2}{2} + \frac{y}{2} \right] dy = \left[\frac{y^4}{8 \cdot 4} - \frac{y^3}{3 \cdot 2} + \frac{y^2}{2 \cdot 2} \right]_0^2$$

$$= \frac{2^4}{8 \cdot 4} - \frac{2^3}{3 \cdot 2} + \frac{2^2}{2 \cdot 2} = \frac{1}{2} - \frac{4}{3} + 1 = \frac{3}{2} - \frac{4}{3} = \frac{1}{6} \quad \checkmark$$

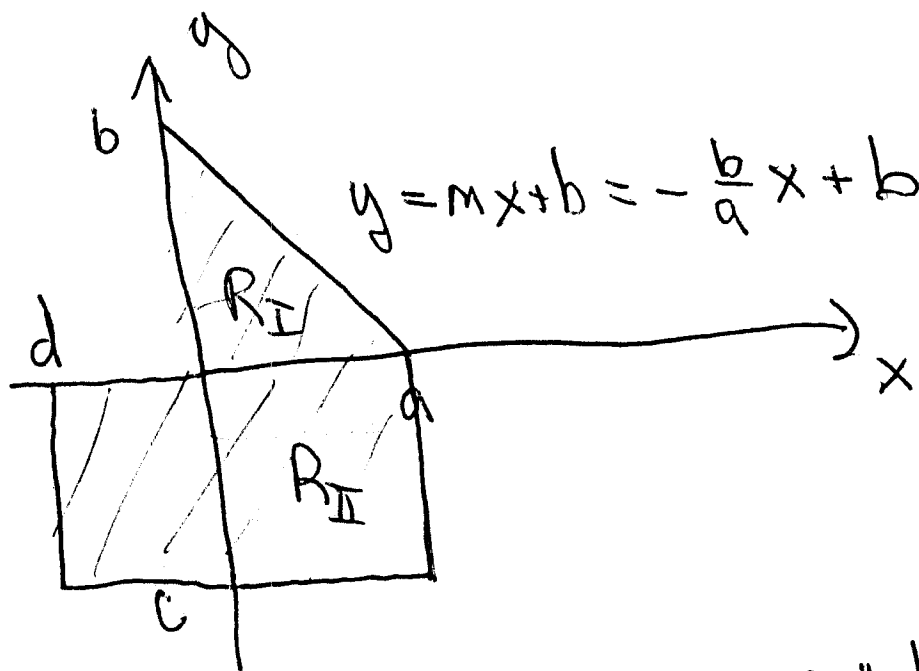
Ex: Some times the region R_{xy} has to be broken up to iterate the integral -

Ex ~~Find~~ set up / iterate the integral

$$\iint_{R_{xy}} f(x,y) dA \quad \text{where } f(x,y) = \tan^{-1}(xy)$$

where:

$$\iint_{R_{xy}} f = \iint_{R_I} f + \iint_{R_{II}} f$$



$$A(y) = \int_0^{y = -\frac{b}{a}x + b} f(x,y) dx \quad \text{in } R_I \quad \iint_{R_I} f dA = \int_0^a \int_0^{-\frac{b}{a}x + b} f(x,y) dx dy$$

$$A(y) = \int_d^a f(x,y) dx \quad \text{in } R_{II} \quad \iint_{R_{II}} f dA = \int_c^0 \int_d^a f(x,y) dx dy$$

①

Theorem Integrals can in general be broken up -

$$\textcircled{1} \iint_{R_I} f(x,y) dA + \iint_{R_{II}} f(x,y) dA = \iint_{R_I \cup R_{II}} f(x,y) dA$$

so long as R_I & R_{II} don't overlap

ie if $R_I \cap R_{II} = \emptyset$

$$\textcircled{2} \iint_{R_{xy}} f(x,y) + g(x,y) dA = \iint_{R_{xy}} f(x,y) dA + \iint_{R_{xy}} g(x,y) dA$$

Eg: $\iint_{R_{xy}} x^2 + \sin xy dA = \iint_{R_{xy}} x^2 dA + \iint_{R_{xy}} \sin xy dA$

$$\textcircled{3} \iint_{R_{xy}} K f(x,y) dA = K \iint_{R_{xy}} f(x,y) dA$$

"you can pull const's thru
I - mean

If $f(x,y) \geq g(x,y) \quad \forall (x,y) \in R_{xy}$, then

(11)

$$\textcircled{4} \quad \iint_{R_{xy}} f(x,y) dA \geq \iint_{R_{xy}} g(x,y) dA$$

Theorem: Fubini I: $\mathbb{R}^2$ $a \leq x \leq b$ $c \leq y \leq d$ (7)

$$\int_a^b \int_c^d f(x,y) dy dx = \iint_R f(x,y) dA = \int_c^d \int_a^b f(x,y) dx dy$$

Ex: We had: $\int_1^3 \int_1^2 xy dy dx = 6$

Check: $\int_1^3 \int_1^2 xy dx dy = \int_1^3 y \left[\int_1^2 x dx \right] dy$

$$= \int_1^3 y \left[\frac{x^2}{2} \right]_1^2 dy = \int_1^3 \left(\frac{4}{2} - \frac{1}{2} \right) y dy$$

$$= \left[\frac{3}{2} \frac{y^2}{2} \right]_1^3 = \frac{3}{2} \left(\frac{9}{2} - \frac{1}{2} \right) = \frac{3}{2} \cdot \frac{8}{2} = \boxed{6}$$



^{Ques.} Radius of gyration - (Of a body about an axis)

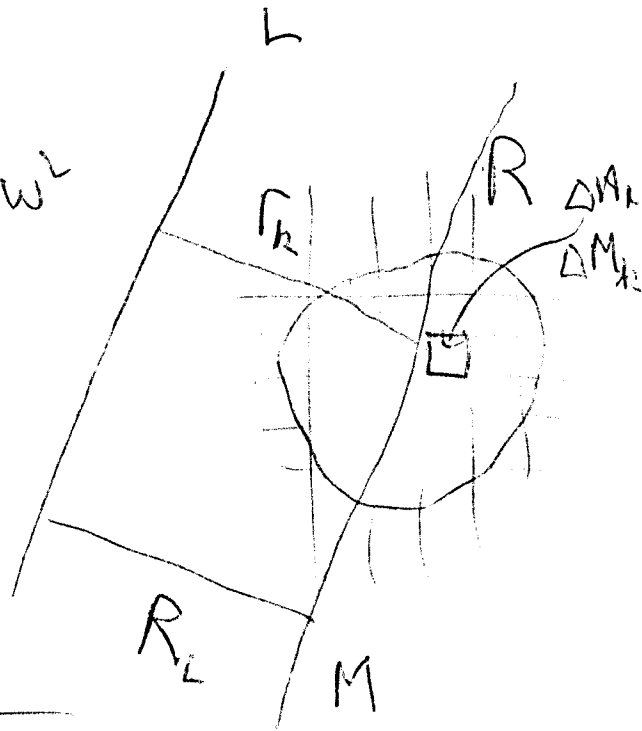
$$R_L = \sqrt{I_L/M}$$

Explain: R_L is radius at which you could put all the mass st you get same KE of rotation.

I.e. want $KE = \frac{1}{2} I_L \omega^2 = \frac{1}{2} M R^2 \omega^2$

$$I_L = M R^2$$

$$\frac{I_L}{M} = R$$



~~Find radius of gyration for above -~~

$$R_y = \sqrt{\frac{I_y}{M}}$$

$$KE = \frac{1}{2} M v^2 = \frac{1}{2} M R^2 \omega^2$$

Ex Draw the region of integration & reverse order:

$$\int_0^{\ln 2} \int_{e^y}^2 dx dy$$

Soln: $\int_{e^y}^2 dx$

$= \int_{x=e^y}^{x=2} dx$

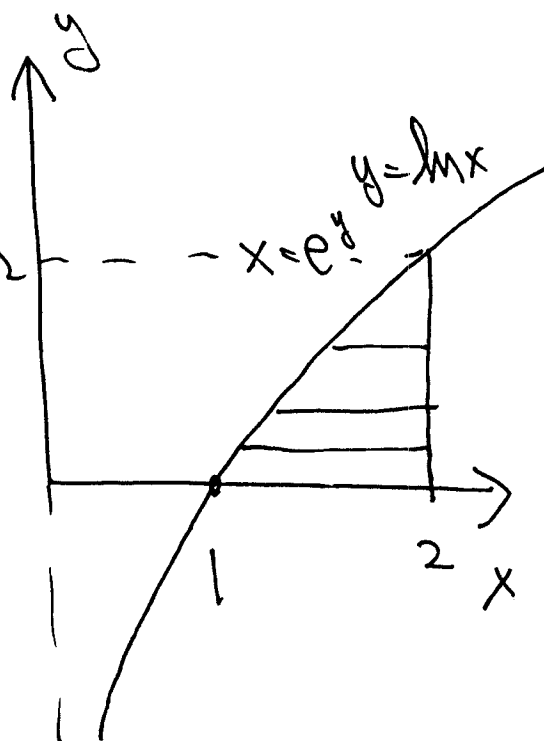
$$\begin{aligned} & \int_0^{\ln 2} (2 - e^y) dy \\ &= 2 \ln 2 - e^y \Big|_0^{\ln 2} \\ &= 2 \ln 2 - e^{\ln 2} + e^0 \\ &= 2 \ln 2 - 2 + 1 \\ &= 2 \ln 2 - 1 \end{aligned}$$

" integrate 1st wrt x from $x=e^y$ to $x=2$

$x=e^y \Leftrightarrow \ln x = y$

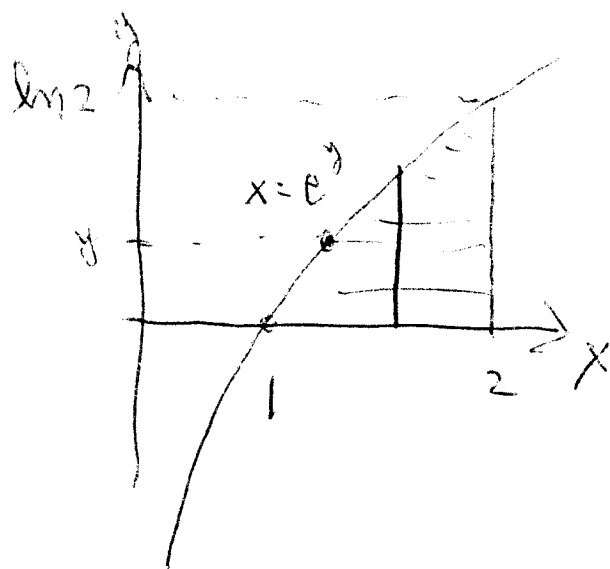
$$\begin{aligned} &= \int_1^2 \int_0^{\ln x} dy dx = \int_1^2 \ln x dx \\ &= x \ln x \Big|_1^2 - \int_1^2 1 dx \\ &= 2 \ln 2 - 2 - 1 = 2 \ln 2 - 1 \end{aligned}$$

$$\begin{aligned} u &= \ln x \\ dv &= dx \\ du &= \frac{1}{x} dx \\ v &= x \end{aligned}$$



$$\int_0^{\ln 2} \int_{x=e^y}^2 dx dy$$

$$x = e^y \Leftrightarrow \ln x = y$$



$$= \int_1^2 \int_{y=0}^{y=\ln x} dy dx = \int_1^2 y \Big|_{y=0}^{y=\ln x} dx = \int_1^2 (\ln x) dx$$

$$\begin{aligned} u &= \ln x \quad dv = dx \\ du &= \frac{dx}{x} \quad v = x \\ &= uv \Big| - \int v du \end{aligned}$$

$$= x \ln x \Big|_1^2 - \int_1^2 dx$$

$$= 2 \ln 2 - 1 \cancel{\ln 1} - 1 = 2 \ln 2 - 1$$

Other way: $\int_0^{\ln 2} \int_{e^y}^2 dx dy = \int_0^{\ln 2} x \Big|_{e^y}^2 dy = \int_0^{\ln 2} (2 - e^y) dy$

$$\begin{aligned} &= 2 \ln 2 - \int_0^{\ln 2} e^y dy = 2 \ln 2 - e^y \Big|_0^{\ln 2} = 2 \ln 2 - 2 + 1 \\ &= 2 \ln 2 - 1 \checkmark \end{aligned}$$